Integrerend project systeemtheorie

23/01/2014, Thursday, 14:00-17:00

You are **NOT** allowed to use any type of calculators.

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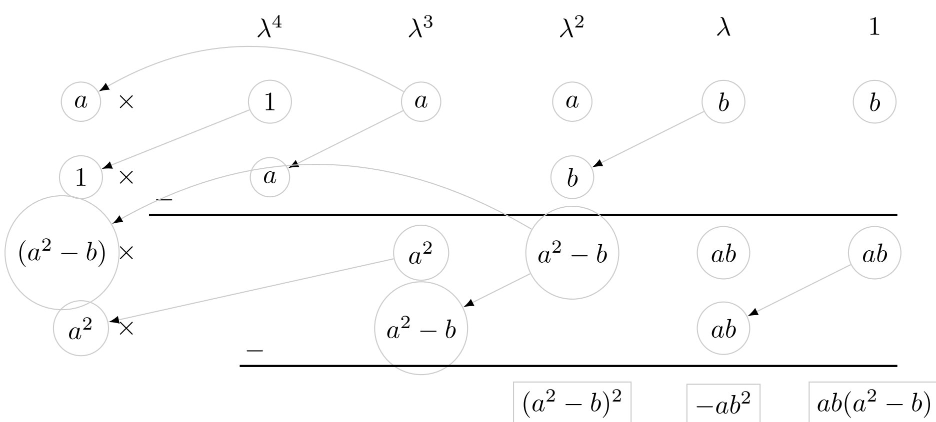
1 (15 pts) **Routh-Hurwitz criterion**

Determine all values $a \in \mathbb{R}$ and $b \in \mathbb{R}$ for which the polynomial $p(\lambda) = \lambda^4 + a\lambda^3 + a\lambda^2 + b\lambda + b$ is stable.

REQUIRED KNOWLEDGE: **Routh-Hurwitz criterion.**

SOLUTION:

Applying Routh-Hurwitz criterion, we get the following table:



From the first reduction step, we see that the polynomial $\lambda^4 + a\lambda^3 + a\lambda^2 + b\lambda + b$ is stable if and only if $a > 0$ and the polynomial $a^2\lambda^3 + (a^2 - b)\lambda^2 + ab\lambda + ab$ is stable. From the second, we see that the polynomial $a^2\lambda^3 + (a^2 - b)\lambda^2 + ab\lambda + ab$ is stable if and only if $a^2(a^2 - b) > 0$ and the polynomial $(a^2 - b)^2\lambda^2 - ab^2\lambda + ab(a^2 - b)$ is stable.

It follows from the first step that $p(\lambda)$ is stable only if $a > 0$. However, this would mean that the polynomial $(a^2 - b)^2\lambda^2 - ab^2\lambda + ab(a^2 - b)$ is not stable. Therefore, $p(\lambda)$ is not stable for all values of a and b.

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Is the system $\dot{x} = Ax$ stable?

(b) Is the system $\dot{x} = Ax + Bu$ controllable?

(c) Is the system $\dot{x} = Ax + Bu$ stabilizable?

(d) Determine the state feedback $u = Fx$ for which the closed loop system matrix $A + BF$ has the characteristic polynomial $p_{A+BF}(\lambda) = (\lambda + 1)^3$.

REQUIRED KNOWLEDGE: **stability, controllability, stabilizability, feedback stabilization.**

SOLUTION:

2a: The system $\dot{x} = Ax$ is stable if and only if all eigenvalues of the matrix A have negative real parts. Note that

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 1 & \lambda \end{bmatrix}\right) = \lambda^3 + \lambda - 1.$$

One can apply the Routh-Hurwitz test to the polynomial $\Delta_A(\lambda) = \lambda^3 + \lambda - 1$ to check if it is stable or not. However, we already know that a monic polynomial is stable only if all its coefficients are positive. Since this is not the case for the polynomial $\Delta_A(\lambda)$, the system is *not* stable.

2b: Note that we have

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

It can be easily verified that the determinant of this matrix is not zero and hence its rank is equal to three. As such, the system is controllable.

2c: Every controllable system is also stabilizable.

2d: From the pole placement theorem, we know that for any monic polynomial $p(\lambda)$ there exists F such that $\Delta_{A+BF}(\lambda) = p(\lambda)$. In order to find the feedback matrix F, we proceed as in the proof of the pole placement theorem. Let

$$q_3 = B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_2 = AB + 0 \cdot B = AB = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$q_1 = A^2B + 0 \cdot AB + 1 \cdot B = A^2B + B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

and also let

$$S = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Note that

$$S^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Let

$$\begin{aligned}
\bar{A} &= S^{-1}AS \\
&= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \\
\bar{B} &= S^{-1}B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Note that $(\lambda+1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1$. Then, we can choose

$$\bar{F} = \begin{bmatrix} -1 & -3 & -3 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -3 \end{bmatrix}.$$

One can verify that $\det(\lambda I - \bar{A} - \bar{B}\bar{F}) = (\lambda+1)^3$. To find F, observe that $\bar{A} + \bar{B}\bar{F} = S^{-1}(A+BF)S$ and hence $F = \bar{F}S^{-1}$. This results in

$$F = \bar{F}S^{-1} = \begin{bmatrix} -2 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -1 \end{bmatrix}$$

Consider the system

$$\dot{x} = \begin{bmatrix} p & q & r \\ 0 & p & q \\ 0 & 0 & p \end{bmatrix} x$$
$$y = \begin{bmatrix} p & q & r \end{bmatrix} x$$

where p, q, and r are real numbers. Determine all values of p, q, and r for which the system

(a) is observable.

(b) is detectable.

Required Knowledge: **eigenvalue test for observability and detectability.**

Solution:

3a: A linear system $\dot{x} = Ax \quad y = Cx$ where $x \in \mathbb{R}^n$ is observable if and only if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \qquad \text{for all } \lambda \in \sigma(A).$$

Since the matrix A of the problem is triangular, the eigenvalues are nothing but the diagonal elements, that is $\sigma(A) = \{p\}$. Then, we have

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & q & r \\ 0 & 0 & q \\ 0 & 0 & 0 \\ p & q & r \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & q & r \\ 0 & 0 & q \\ 0 & 0 & 0 \\ p & 0 & 0 \end{bmatrix} = 3 \quad \Longleftrightarrow \quad p \neq 0 \neq q.$$

3b: A linear system $\dot{x} = Ax \quad y = Cx$ where $x \in \mathbb{R}^n$ is detectable if and only if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \qquad \text{for all } \lambda \in \sigma(A) \text{ with } \operatorname{Re}(\lambda) \geqslant 0.$$

Then, we can conclude that the system we have is detectable if and only if

$$p < 0 \text{ OR } (\, p > 0 \text{ AND } q \neq 0\,).$$

Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$. Show that the subspace

$$\langle \ker C \mid A \rangle := \ker C \cap \ker CA \cap \cdots \cap \ker CA^{n-1}$$

is the largest A-invariant subspace that is contained in $\ker C$.

REQUIRED KNOWLEDGE: **invariance under a linear map.**

SOLUTION:
Note first that the subspace $\langle \ker C \mid A \rangle$ is contained in $\ker C$ by definition. To show that it is A-invariant, let $x \in \langle \ker C \mid A \rangle$. This would mean that

$$CA^k x = 0$$

for $k = 0, 1, \ldots, n-1$. Then, it follows from the Cayley-Hamilton theorem that

$$CA^k x = 0$$

for all $k \geqslant 0$. In turn, this implies that $Ax \in \langle \ker C \mid A \rangle$. Hence, $\langle \ker C \mid A \rangle$ is A-invariant. In order to show that it is the largest of such subspaces, let $\mathcal{V}$ be an A-invariant subspace that is contained in $\ker C$. Then, we have

$$A\mathcal{V} \subseteq \mathcal{V} \subseteq \ker C.$$

By repeating this argument, we get

$$A^2\mathcal{V} \subseteq A\mathcal{V} \subseteq \mathcal{V} \subseteq \ker C.$$

Therefore, one can conclude by induction on k that

$$A^k\mathcal{V} \subseteq \ker C$$

for all $k \geqslant 0$. Equivalently, we have

$$\mathcal{V} \subseteq \ker CA^k$$

for all $k \geqslant 0$. Hence, we see that $\mathcal{V} \subseteq \langle \ker C \mid A \rangle$. In other words, $\langle \ker C \mid A \rangle$ is the largest A-invariant subspace that is contained in $\ker C$.